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A STRONG LAW OF LARGE NUMBERS FOR NONEXPANSIVE VECTOR-VALUED STOCHASTIC PROCESSES*

BY

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ABSTRACT

A map $T\colon X \to X$ on a normed linear space is called **nonexpansive** if $\|Tx - Ty\| \le \|x - y\|$ $\forall x, y \in X$. Let (Ω, Σ, P) be a probability space, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ an increasing chain of σ-fields spanning Σ, X a Banach space, and $T\colon X \to X$. A sequence $(\mathbf{x}_n)$ of strongly $\mathcal{F}_n$-measurable and strongly P-integrable functions on Ω taking on values in X is called a **T-martingale** if $E(\mathbf{x}_{n+1} \mid \mathcal{F}_n) = T(\mathbf{x}_n)$.

Let $T\colon H \to H$ be a nonexpansive mapping on a Hilbert space H, and let $(\mathbf{x}_n)$ be a T-martingale taking on values in H. If

$$\sum_{n=1}^{\infty} n^{-2} E\|\mathbf{x}_{n+1} - T\mathbf{x}_n\|^2 < \infty,$$

then $\mathbf{x}_n/n$ converges a.e.

Let $T\colon X \to X$ be a nonexpansive mapping on a p-uniformly smooth Banach space X, $1 < p \le 2$, and let $(\mathbf{x}_n)$ be a T-martingale (taking on values in X). If

$$\sum n^{-p} E(\|\mathbf{x}_n - T\mathbf{x}_{n-1}\|^p) < \infty,$$

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then there exists a continuous linear functional $f \in X^*$ of norm 1 such that

$$\lim_{n\to\infty} f(\mathbf{x}_n)/n = \lim_{n\to\infty} \|\mathbf{x}_n\|/n = \inf\{\|Tx - x\| : x \in X\} \text{ a.e.}$$

If, in addition, the space X is strictly convex, $\mathbf{x}_n/n$ converges weakly; and if the norm of X^* is Fréchet differentiable (away from zero), $\mathbf{x}_n/n$ converges strongly.

1. Introduction

The Operator Ergodic Theorem (OET) asserts that, if A: $H \to H$ is a linear operator with norm 1 on a Hilbert space, then, for every $x \in H$,

$$\frac{x + Ax + \cdots + A^n x}{n} \text{ converges (strongly).}$$

The Strong Law of Large Numbers (SLLN) for martingales in Hilbert spaces says that if $(\mathbf{x}_n)$ is an H-valued martingale such that

$$\sum_{k=1}^{\infty} k^{-2} E(\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2) < \infty,$$

then

$$\frac{\mathbf{x}_n}{n} \text{ converges a.e. (to zero).}$$

In the proposition below, we provide a result that generalizes both these classical theorems.

A map T: $X \to X$ on a normed linear space is called **nonexpansive** if $\|Tx - Ty\| \le \|x - y\|$ $\forall x, y \in X$.

Let (Ω, Σ, P) be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n$ be an increasing chain of σ-fields spanning Σ. A sequence $(\mathbf{x}_n)$ of strongly $\mathcal{F}_n$-measurable and strongly P-integrable functions on Ω taking on values in a (real separable*) Banach space X, is called an X-**valued stochastic process.** If, in addition, for some map T: $X \to X$,

$$E(\mathbf{x}_{n+1} \mid \mathcal{F}_n) = T(\mathbf{x}_n), \quad n = 0, 1, \ldots,$$

* The results hold for any Banach space. However, as the values of any sequence $(\mathbf{x}_n)$ of strongly $\mathcal{F}_n$-measurable and strongly P-integrable functions on Ω taking on values in a Banach space are with probability 1 in a separable subspace, we may assume w.l.o.g. that the values are in a separable B-space.

then $(\mathbf{x}_n)$ is called a T-**martingale**.

Of course, if T is the identity, then T-martingales are just ordinary martingales. In general, the class of all T-martingales consists of all sequences $(\mathbf{x}_n)$ of the form $\mathbf{x}_0 = \mathbf{d}_0, \ldots, \mathbf{x}_{n+1} = T(\mathbf{x}_n) + \mathbf{d}_{n+1}$ where $(\mathbf{d}_n)$ is an ordinary martingale-difference sequence, i.e., $E(\mathbf{d}_{n+1} \mid \mathcal{F}_n) = 0$.

PROPOSITION 1: *Let $T\colon H \to H$ be a nonexpansive mapping on a Hilbert space H, and let $(\mathbf{x}_n)$ be a T-martingale taking on values in H. If*

$$\sum_{n=1}^{\infty} n^{-2} E(\|\mathbf{x}_{n+1} - T\mathbf{x}_n\|^2) < \infty,$$

then

$$\frac{\mathbf{x}_n}{n} \text{ converges a.e.}$$

To see that the proposition in fact includes both the SLLN and the OET (for Hilbert spaces), note the following equivalent reformulation of the OET: *If $T\colon H \to H$ is a nonexpansive affine mapping on a Hilbert space, then $T^n x/n$ converges $\forall x \in H$.*

To verify the equivalence of the formulations note that any map $T\colon H \to H$ is a nonexpansive affine map iff it is of the form $Ty = x + Ay$ where A is a linear operator of norm less than or equal to one; since $T^n y = x + Ax + \ldots + A^{n-1}x + A^n y$, the sequence $T^n x/n$ converges $\forall x \in H$ iff the sequence $(x + Ax + \cdots + A^{n-1}x)/n$ converges $\forall x \in H$.

Thus the OET can be obtained from the proposition by restricting attention to deterministic $(\mathbf{x}_n)$, whereas the SLLN is the special case where T is the identity.

But the proposition also yields results combining the OET and the SLLN. For example, we show that it implies the following.

If $A\colon H \to H$ is a linear operator of norm 1 on a Hilbert space, and if $B_i\colon H \to H$ are (random) linear operators of norm at most 1 such that

$$E(B_n \mid B_1, \ldots, B_{n-1}) = A$$

and

$$\sum_{k=1}^{\infty} E(\|B_k - A\|^2) < \infty,$$

then, for every $x \in H$, almost everywhere

$$\lim_{n\to\infty} A_n x = \lim_{n\to\infty} \frac{x + Ax + A^2x + \cdots + A^n x}{n+1}$$

where

$$A_n = \frac{I + B_n + B_nB_{n-1} + \cdots + B_nB_{n-1}\cdots B_1}{n+1}.$$

In the next section, we present the general version of Proposition 1, which encompasses more general versions of the SLLN (e.g. Woyczyński [6] and Hoffmann-Jorgensen and Pisier [4]) and of the OET.

2. The main result

Before stating our theorem we review some definitions.

Given a Banach space X we denote by $S(X)$ the set of all vectors $x \in X$ with $\|x\| = 1$, where X^* denotes the dual of X.

A Banach space X is **strictly convex** if

$$\|x + y\| < 2 \quad \forall x, y \in S(X) \quad \text{with } x \neq y.$$

The modulus of smoothness of a Banach space X is the function $\rho_X\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $\rho_X(t) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1 : \|x\| = 1 \text{ and } \|y\| \leq t\}$. X is **uniformly smooth** if $\rho_X(t) = o(t)$ as $t \to 0+$; it is p-**uniformly smooth**, $1 < p \leq 2$, if $\rho_X(t) = O(t^p)$ as $t \to 0+$.

The norm of a Banach space X is **Fréchet differentiable** (away from zero) whenever for every $x \in X$ with $x \neq 0$, $\lim_{\lambda\to 0}(\|x+\lambda y\| - \|x\|)/\lambda$ exists uniformly in $y \in S(X)$.

To simplify the statement below, we define a Banach space to be 1-**uniformly smooth** if it is uniformly smooth*.

THEOREM 1: *Let T: $X \to X$ be a nonexpansive mapping on a p-uniformly smooth Banach space X, $1 \leq p \leq 2$, and let $(\mathbf{x}_n)$ be a T-martingale (taking on values in X). If*

$$\sum n^{-p} E(\|\mathbf{x}_n - T\mathbf{x}_{n-1}\|^p) < \infty, \tag{1}$$

then there exists a continuous linear functional $f \in S(X^*)$ such that

$$\lim_{n\to\infty} \frac{f(\mathbf{x}_n)}{n} = \lim_{n\to\infty} \frac{\|\mathbf{x}_n\|}{n} = \inf\{\|Tx - x\| : x \in X\} \quad \text{a.e.} \tag{2}$$

If, in addition, the space X is strictly convex,

$$\mathbf{x}_n/n \text{ converges weakly to a point in } X; \tag{3}$$

* Note that if X is p-uniformly smooth for some $1 \leq p \leq 2$, then it is uniformly smooth and thus (Diestel [2, p. 38]) reflexive.

and if the norm of X^* is Fréchet differentiable (away from zero),

(4) $\mathbf{x}_n/n$ *converges strongly to a point in* X.

Proposition 1 is a special case of the theorem because any Hilbert space, H, is 2-uniformly smooth, and the norm of H^* (i.e., H) is Fréchet differentiable. Hoffmann-Jorgensen and Pisier [4] demonstrate the SLLN for martingales in a p-uniformly smooth Banach space, under condition (1). Thus, Theorem 1 may be viewed as a generalization of both the Hoffmann-Jorgensen and Pisier SLLN for martingales and the OET for p-uniformly smooth Banach spaces.

When $(\mathbf{x}_n)$ is a deterministic sequence, the conclusions of the theorem already follow from the nonexpansiveness of T and the reflexivity of X (which is weaker than the p-uniform smoothness of X) and assumption (1) is obviously redundant. In fact conclusions (2), (3) and (4) are Theorem 1.1, and Corollaries 1.3 and 1.2 of Kohlberg and Neyman [5], respectively.

The extension of those results to the stochastic case requires the stronger conditions of Theorem 1. Indeed, we show that weaker conditions do not suffice: If the norm of X is not Fréchet differentiable, we can construct a nonexpansive T-martingale $(\mathbf{x}_n)$ satisfying $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq 1$ everywhere and for which $\liminf \|\mathbf{x}_n\|/n < \limsup \|\mathbf{x}_n\|/n$.

One may wonder whether weaker conditions would guarantee that x_n converges in direction, i.e., that $x_n/\|x_n\|$ converges: We construct an example of a finite dimensional normed space X that is not smooth and a T-martingale $(\mathbf{x}_n)$ satisfying $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq 1$ and $\liminf \|\mathbf{x}_n\|/n > 0$, yet $\mathbf{x}_n/\|\mathbf{x}_n\|$ does not converge.

3. Preliminaries

The norm of a Banach space X is uniformly smooth whenever $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $\forall x \in S(X)$ and $\forall\|y\| \leq \delta$,

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|.$$

If X is p-uniformly smooth, $1 \leq p \leq 2$, then X is uniformly smooth.

X is uniformly smooth iff every support mapping $x \to f_x$ is norm-norm continuous from $S(X)$ to $S(X^*)$ (e.g., Diestel [2, p. 36]).

The property of p-uniformly smooth spaces that is crucial for Theorem 1 is that for any point, x, on the unit sphere, the norm in any neighboring point $x + y$ is approximated by the supporting linear functional at x up to an error of order $\|y\|^p$. Formally, LEMMA 1: *Let ρ be the modulus of smoothness of X. Then for all $x, y \in X$, with $x \neq 0$,*

$$\|x+y\| \leq f_x(x+y) + 2\rho(\|y\|/\|x\|)\|x\|,$$

where $f_x \in S(X^*)$ with $f_x(x) = \|x\|$.

Proof: By the definition of the modulus of smoothness,

$$\left\|\frac{x}{\|x\|} + \frac{y}{\|x\|}\right\| \leq 2 - \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| + 2\rho(\|y\|/\|x\|),$$

and therefore by multiplying both sides of the inequality by $\|x\|$ and using the equality $2\|x\| - f_x(x-y) = f_x(x+y)$, we have

$$\|x+y\| \leq f_x(x+y) + 2\rho(\|y\|/\|x\|)\|x\|. \qquad \blacksquare$$

4. Proof of the main result

It is easy to verify (see Diestel [2]) the following geometric interpretation of our conditions: X is strictly convex and reflexive* (respectively, the norm of X^* is Fréchet differentiable) iff every sequence $x_n \in S(X)$ satisfying $f(x_n) \to 1$ for some $f \in S(X^*)$ converges weakly (respectively, strongly).

Thus, conclusions (3) and (4) of the theorem are immediate consequences of (2), which is restated below:

PROPOSITION 2: *Let X be a p-uniformly smooth Banach space, $1 \leq p \leq 2$, and let T: $X \to X$ be a nonexpansive mapping. Then for every T-martingale, $(\mathbf{x}_n)$, satisfying*

$$\sum n^{-p} E(\|\mathbf{x}_{n+1} - T\mathbf{x}_n\|^p) < \infty,$$

there is a linear functional $f \in S(X^*)$ such that

$$\lim_{n\to\infty} \frac{f(\mathbf{x}_n)}{n} = \lim_{n\to\infty} \frac{\|\mathbf{x}_n\|}{n} = \inf\{\|Tx - x\| : x \in X\}, \text{ a.e.}$$

Proof: Let α denote $\inf_{x\in X} \|Tx - x\|$ and assume first that $\alpha > 0$. By lemma 2.3 of [5], for all $r > 0$ and x in X,

$$\|x(r) - Tx\| \leq \|x(r) - x\| - \alpha + 2r\|Tx\|, \tag{5}$$

where $x(r)$ is the unique fixed point of the contraction $T/(1+r)$. Note that $T(x(r)) - x(r) = rx(r)$ and thus $\|rx(r)\| \geq \alpha$, which in particular implies that

* In the context of the theorem, reflexivity is automatically satisfied because p-uniformly smooth spaces, $1 \leq p \leq 2$, are reflexive.

$\|x(r)\| \to \infty$ as $r \to 0+$. For each y in X, $y \neq 0$, we denote by f_y the linear functional of norm 1 (in X^*) satisfying $f_y(y) = \|y\|$. (That such a linear functional exists follows from the Hahn–Banach theorem; its uniqueness follows from the differentiability of the norm.) Fix $x \in X$. Since $\|x(r)\| \to \infty$ as $r \to 0+$, $x(r) - x \neq 0$ for sufficiently small r and thus $f_{x(r)-x}$ is well-defined. From (5), the inequality $f_{x(r)-x}(x(r) - Tx) \leq \|x(r) - Tx\|$, and the equality $f_{x(r)-x}(x(r) - x) = \|x(r) - x\|$, it follows that

$$f_{x(r)-x}(Tx - x) \geq \alpha - 2r\|Tx\|.$$

Let f be a limit point of the linear functionals $f_{x(r)-x}$ (as $r \to 0+$) in the weak*-topology with $\|f\| \leq 1$ (the existence of such an f is guaranteed by the Banach–Alaoglu Theorem). Then $f_{x(r)-x}(Tx - x) \to f(Tx - x)$ as $r \to 0+$ and $2r\|Tx\| \to 0$ as $r \to 0+$, and thus

$$f(Tx - x) \geq \alpha.$$

As the norm of X is p-uniformly smooth, it is in particular uniformly smooth, which implies that the support mapping $z \to f_z$ from $S(X) = \{x \in X : \|x\| = 1\}$ to X^* is norm-norm uniformly continuous. As $\|x(r)\| \to \infty$, for all x, y in X, $\|(x(r) - x)/\|x(r) - x\| - (x(r) - y)/\|x(r) - y\|\| \to 0$ as $r \to \infty$ and thus f is also a w^*-limit point of $f_{x(r)-y}$ as $r \to 0+$ and therefore

$$f(Ty - y) \geq \alpha \quad \text{for all } y \in X. \tag{6}$$

As $\inf\{\|Ty - y\| : y \in X\} = \alpha > 0$, the inequalities $\alpha \leq f(Tx - x) \leq \|f\|\|Tx - x\|$ imply that $\|f\| \geq 1$, and thus $\|f\| = 1$. Altogether we have shown the existence of f in X^* with $\|f\| = 1$ for which

$$f(E(\mathbf{x}_{n+1}|\mathcal{F}_n)) \geq f(\mathbf{x}_n) + \alpha.$$

As f is linear, $f(T\mathbf{x}_n) = f(E(\mathbf{x}_{n+1}|\mathcal{F}_n)) = E(f(\mathbf{x}_{n+1})|\mathcal{F}_n)$ and thus

$$f(T\mathbf{x}_n) = E(f(\mathbf{x}_{n+1})|\mathcal{F}_n) \geq f(\mathbf{x}_n) + \alpha. \tag{7}$$

Set $\mathbf{z}_n = f(\mathbf{x}_n) - E(f(\mathbf{x}_n)|\mathcal{F}_{n-1})$. Then $\mathbf{z}_n$ is an $(\mathcal{F}_n)$-adapted stochastic process with $E(\mathbf{z}_n|\mathcal{F}_{n-1}) = 0$. The inequality $|f(\mathbf{x}_n) - E(f(\mathbf{x}_n) \mid \mathcal{F}_{n-1})| \leq \|\mathbf{x}_n - T\mathbf{x}_{n-1}\|$ implies that $\sum_{n=1}^{\infty} n^{-p} E(|\mathbf{z}_n|^p) < \infty$ and therefore, by the strong law of large numbers for real-valued martingales (see e.g. Feller [3], II, p. 243 for $p = 2$, and Chow [1] for $1 < p$), $n^{-1}\sum_{k=1}^{n} \mathbf{z}_k \to 0$, as $n \to \infty$, a.e. Recall (see (7)) that

$E(f(\mathbf{x}_k)|\mathcal{F}_{k-1}) \geq f(\mathbf{x}_{k-1}) + \alpha$ and so $\mathbf{z}_k \leq f(\mathbf{x}_k) - f(\mathbf{x}_{k-1}) - \alpha$, and thus by summing over $k = 2, \ldots, n$, $\sum_{k=1}^n \mathbf{z}_k \leq f(\mathbf{x}_n) - f(\mathbf{x}_1) - (n-1)\alpha$, which implies that $\liminf_{n\to\infty} n^{-1}(f(\mathbf{x}_n) - n\alpha) \geq \liminf_{n\to\infty} n^{-1} \sum_{k=1}^n \mathbf{z}_k = 0$ a.e. and thus

$$\liminf_{n\to\infty} f(\mathbf{x}_n)/n \geq \alpha \quad \text{a.e.} \tag{8}$$

Fix $\varepsilon > 0$ and let $x^\varepsilon \in X$ be such that $\|Tx^\varepsilon - x^\varepsilon\| < \alpha + \varepsilon$. Set $\mathbf{x}_k^\varepsilon = \mathbf{x}_k - x^\varepsilon$, and $T^\varepsilon x = Tx - x^\varepsilon$.

For every k let f_k be a random variable with values in $S(X^*)$ and such that $f_k(T^\varepsilon \mathbf{x}_k) = \|T^\varepsilon \mathbf{x}_k\|$. Denote $I_k = I(f(\mathbf{x}_k^\varepsilon) > \alpha k/2)$ and observe that $\|T^\varepsilon \mathbf{x}_k\| \geq f(T^\varepsilon \mathbf{x}_k) \geq f(\mathbf{x}_k^\varepsilon)$ and thus $\|T^\varepsilon \mathbf{x}_k\| \geq \alpha k/2$ on $I_k = 1$.

Define $\mathbf{z}_{k+1} = f_k(\mathbf{x}_{k+1}^\varepsilon) - \|T^\varepsilon \mathbf{x}_k\|$. Then $\mathbf{z}_k$ is an $\mathcal{F}_k$-adapted stochastic process of martingale differences. As $|z_{k+1}| \leq \|\mathbf{x}_{k+1} - T\mathbf{x}_k\|$,

$$\sum_{k=1}^\infty k^{-p} E(|\mathbf{z}_{k+1}|^p) \leq \sum_{k=1}^\infty k^{-p} E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p) < \infty \quad \text{a.e.},$$

and therefore by the strong law of large numbers for martingale differences (Chow [1]),

$$\frac{1}{n}\sum_{k=1}^n f_k(\mathbf{x}_{k+1}^\varepsilon) - \|T^\varepsilon \mathbf{x}_k\| \to 0 \quad \text{as } n \to \infty \quad \text{a.e.} \tag{9}$$

By Lemma 1, on $T^\varepsilon \mathbf{x}_k \neq 0$,

$$\|\mathbf{x}_{k+1}^\varepsilon\| - f_k(\mathbf{x}_{k+1}^\varepsilon) \leq 2\rho\left(\frac{\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|}{\|T^\varepsilon \mathbf{x}_k\|}\right)\|T^\varepsilon \mathbf{x}_k\|.$$

As X is p-uniformly smooth, there is a positive constant C such that

$$2\rho\left(\frac{\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|}{\|T^\varepsilon \mathbf{x}_k\|}\right)\|T^\varepsilon \mathbf{x}_k\| \leq C\frac{\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p}{\|T^\varepsilon \mathbf{x}_k\|^{p-1}}$$

on $T^\varepsilon \mathbf{x}_k \neq 0$, and therefore as $\|T^\varepsilon \mathbf{x}_k\| \geq \alpha k/2$ on I_k,

$$(\|\mathbf{x}_{k+1}^\varepsilon\| - f_k(\mathbf{x}_{k+1}^\varepsilon))I_k \leq K\frac{\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p}{k^{p-1}}.$$

By assumption, $\sum_{k=1}^\infty k^{-p} E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p)$ is finite a.e. and therefore $\sum_{k=1}^\infty k^{-p}\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p$ is finite a.e. and thus, by Kronecker's Lemma,

$$\frac{1}{n}\sum_{k=1}^n k^{1-p}\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p \underset{n\to\infty}{\to} 0 \quad \text{a.e.} \tag{10}$$

Therefore, a.e.,

$$\frac{1}{n}\sum_{k=1}^{n}(\|\mathbf{x}_{k+1}^{\varepsilon}\| - f_k(\mathbf{x}_{k+1}^{\varepsilon}))I_k \to 0 \quad \text{as } n \to \infty.$$

As $\alpha > 0$ by assumption, (8) implies that almost everywhere, $I(f(\mathbf{x}_k^{\varepsilon}) > \alpha k/2) = 1$ for sufficiently large k, and therefore

$$\frac{1}{n}\sum_{k=1}^{n}(\|\mathbf{x}_{k+1}^{\varepsilon}\| - f_k(\mathbf{x}_{k+1}^{\varepsilon})) \to 0 \quad \text{as } n \to \infty,$$

which, together with (9), implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|\mathbf{x}_{k+1}^{\varepsilon}\| - \|T^{\varepsilon}\mathbf{x}_k\| = 0. \tag{11}$$

By the nonexpansiveness of T, $\|T\mathbf{x}_k - Tx^{\varepsilon}\| \leq \|\mathbf{x}_k - x^{\varepsilon}\|$, and therefore, by the triangle inequality,

$$\|T^{\varepsilon}\mathbf{x}_k\| \leq \|\mathbf{x}_k^{\varepsilon}\| + \|Tx^{\varepsilon} - x^{\varepsilon}\| \leq \|\mathbf{x}_k^{\varepsilon}\| + \alpha + \varepsilon,$$

and therefore

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|T^{\varepsilon}\mathbf{x}_k\| - \|\mathbf{x}_k^{\varepsilon}\| \leq \alpha + \varepsilon,$$

which, together with (11) and the equality $\lim_{n\to\infty}(\|\mathbf{x}_n^{\varepsilon}\| - \|\mathbf{x}_n\|)/n = 0$, imply that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|\mathbf{x}_{k+1}^{\varepsilon}\| - \|\mathbf{x}_k^{\varepsilon}\| = \limsup_{n\to\infty}\frac{\|\mathbf{x}_{n+1}\|}{n} - \frac{\|\mathbf{x}_1\|}{n} \leq \alpha + \varepsilon.$$

As this holds for all $\varepsilon > 0$ we deduce that

$$\limsup\frac{\|\mathbf{x}_n\|}{n} \leq \alpha, \tag{12}$$

which, together with (8), implies that

$$\lim\frac{\|\mathbf{x}_n\|}{n} = \alpha = \lim\frac{f(\mathbf{x}_n)}{n} \quad \text{a.e.}$$

which completes the proof in the case that $\alpha > 0$.

If $\alpha = 0$, let $Y = (X \oplus \mathbb{R})_2$; i.e., Y is the direct sum of X and $\mathbb{R}$ with the norm $\|(x,a)\| = (\|x\|^2 + a^2)^{1/2}$. Then Y is p-uniformly smooth. Define the Y-valued stochastic process $(\mathbf{y}_n)_{n=1}^{\infty}$ by

$$\mathbf{y}_n = (\mathbf{x}_n, n).$$

Then $E(\mathbf{y}_{n+1} \mid \mathcal{F}_n) = (T\mathbf{x}_n, n+1) = S(\mathbf{y}_n)$ where S: $Y \to Y$ is given by $S(x,a) = (Tx, a+1)$. Note that S is nonexpansive, $\inf\{\|Sy - y\| : y \in Y\} = 1$, and that $\|\mathbf{y}_{n+1} - S\mathbf{y}_n\| = \|\mathbf{x}_{n+1} - \mathbf{x}_n\|$ and therefore $\sum n^{-p} E(\|\mathbf{y}_{n+1} - S\mathbf{y}_n\|^p) < \infty$. Thus by the already proved result, we have that a.e.

$$1 = \lim_{n\to\infty} \frac{\|\mathbf{y}_n\|}{n} = \lim_{n\to\infty} \frac{(\|\mathbf{x}_n\|^2 + n^2)^{1/2}}{n} = \lim_{n\to\infty} (1 + \|\mathbf{x}_n\|^2/n^2)^{1/2}.$$

Therefore, $\|\mathbf{x}_n\|^2/n^2 \to 0$ a.s. as $n \to \infty$, which implies that

$$\lim_{n\to\infty} \frac{\|\mathbf{x}_n\|}{n} = 0 \quad \text{a.e.} \qquad \blacksquare$$

5. Additional results

We start with a corollary that is a special case of Theorem 1.

COROLLARY 1: *Let A: $X \to X$ be a linear operator of norm 1 on a 2-uniformly smooth Banach space X. Let B_k, $k = 1, \ldots,$ be an $\mathcal{F}_k$-adapted stochastic process of linear operators on X such that*

$$E(B_n \mid \mathcal{F}_{n-1}) = A$$

and

$$\sum_{k=1}^{\infty} E(\|B_k - A\|^2) < \infty.$$

If there is a constant K such that $\|A_n\| \le K$ a.e. where A_n is the random linear operator

$$A_n = \frac{I + B_n + B_n B_{n-1} + \cdots + B_n B_{n-1} \cdots B_1}{n+1},$$

then, for every $x \in X$, there is a linear functional $f \in S(X^*)$ such that

$$\lim_{n\to\infty} f(A_n x) = \lim_{n\to\infty} \|A_n x\| = \inf\{\|x + Ay - y\| : y \in X\} \quad \text{a.e.}$$

If, in addition, the space X is strictly convex,

(13) $\qquad$ *$A_n x$ converges weakly to a point in X;*

and if the norm of X^* is Fréchet differentiable (away from zero),

(14) $\qquad A_n x$ *converges strongly to a point in* X.

Proof: Define the following T-martingale. $T\colon H \to H$ is given by $Ty = x + Ay$, $x_1 = x$ and $x_{n+1} = x + B_n x_n$. Thus $\mathbf{x}_{n+1} = A_n \mathbf{x}$, and, as $\|A_n\| \leq K$, $k^{-2}\|x_{k+1} - Tx_k\|^2 \leq K\|B_k - A\|^2$ and therefore

$$\sum_{k=1}^{\infty} k^{-2} E(\|x_{k+1} - Tx_k\|^2) < \infty,$$

and thus by Theorem 1 there exists $f \in S(X^*)$ with

$$\lim_{n\to\infty} f(A_n x) = \lim_{n\to\infty} \|A_n x\| = \inf\{\|x + Ay - y\| : y \in X\} \quad \text{a.e.}$$

Conclusions (13) and (14) follow as in the proof of Theorem 1. ∎

The following example due to H. Furstenberg and Y. Katznelson illustrates that the uniform boundedness of the $\|A_k\|$ does not follow from the assumption $\sum_{k=1}^{\infty} E(\|B_k - A\|^2) < \infty$ even when the B_is are independent: let f_k be the sequence of ± 1 valued Rademacher functions. Consider the sequence of independent random operators B_i defined on $H = L_2[0,1]$ by $B_i g = (1 + f_k/k^{2/3})g$ with probability $1/2$ and $B_i g = (1 - f_k/k^{2/3})g$ with probability $1/2$. The random operators B_i, $i = 1, 2, \ldots$, satisfy $E(B_n \mid B_1, \ldots, B_{n-1}) = I$ and $E(\|B_i - I\|^2) = i^{-2/3}$ and thus $\sum_{k=1}^{\infty} \|B_i - I\|^2 < \infty$. The norm of the operator A_n equals $(1 + (1 + 1/n^{2/3}) + \cdots + (1 + 1/n^{2/3}) \cdots (1 + 1/1^{2/3}))/(n+1)$, which is at least $\exp \sum_{k=1}^{n} k^{-2/3}/(n+1)$, which converges to infinity as n goes to ∞; thus the sequence $\|A_p\|$ is unbounded and, by the uniform boundedness theorem, there is $g \in H$ such that the sequence $A_k g$, $k = 1, 2 \ldots$ is unbounded and thus obviously does not converge.

A Banach space that is p-uniformly smooth, $1 \leq p \leq 2$, is uniformly smooth and reflexive. Our next result gives convergence results for $\|\mathbf{x}_n\|/n$ (and $\mathbf{x}_n/n$) in uniformly smooth (and reflexive) spaces under an alternative assumption to (1).

THEOREM 2: *Let $T\colon X \to X$ be a nonexpansive mapping on a uniformly smooth Banach space X, and let $(\mathbf{x}_n)$ be a T-martingale (taking on values in X). If*

(15) $$E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^2 \mid \mathcal{F}_k) \leq V,$$

then there exists a continuous linear functional $f \in S(X^*)$ such that

(16) $$\lim_{n\to\infty} \frac{f(\mathbf{x}_n)}{n} = \lim_{n\to\infty} \frac{\|\mathbf{x}_n\|}{n} = \inf\{\|Tx - x\| : x \in X\} \quad \text{a.e.}$$

If, in addition, the space X is strictly convex,

$$\text{(17)} \qquad \mathbf{x}_n/n \text{ converges weakly to a point in } X;$$

and if the norm of X^* is Fréchet differentiable (away from zero),

$$\text{(18)} \qquad \mathbf{x}_n/n \text{ converges strongly to a point in } X.$$

Proof: As in the proof of Theorem 1, it is enough to prove (16); (17) and (18) follow. Assume that $\alpha \equiv \inf\{\|Tx - x\| : x \in X\} > 0$. By the proof of Proposition 2, it follows that if X is uniformly smooth there is an $f \in S(X^*)$ with $f(Ty) - f(y) \geq \alpha$ for every $y \in X$ and

$$\text{(19)} \qquad \liminf f(\mathbf{x}_n)/n \geq \alpha.$$

It is therefore sufficient to prove that

$$\text{(20)} \qquad \limsup \|\mathbf{x}_n\|/n \leq \alpha.$$

Together they imply that $\|\mathbf{x}_n\|/n$ converges.

Let $V \geq E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^2 \mid \mathcal{F}_k)$. Fix $\varepsilon > 0$ and let $K > 0$ be sufficiently large so that $V/K < \varepsilon$. Then, as $KE(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| > K)|\mathcal{F}_k) \leq E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^2|\mathcal{F}_k) \leq V$,

$$\text{(21)} \qquad E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| > K)|\mathcal{F}_k) < \varepsilon,$$

and as $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq \|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^2 + 1$,

$$\text{(22)} \qquad E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|/(V+1)|\mathcal{F}_k) \leq 1.$$

As X is uniformly smooth, there is an $M > 0$ such that if $z \in S(X)$ and $y \in X$ with $\|y - z\| < K/M$, then

$$\|y\| \leq f_z(y) + \varepsilon\|y - z\|/(V+1).$$

Thus, by choosing x^ε in X with $\|Tx^\varepsilon - x^\varepsilon\| \leq \alpha + \varepsilon$, setting $\mathbf{x}_k^\varepsilon = \mathbf{x}_k - x^\varepsilon$, and using the above inequality (with $z = T\mathbf{x}_k - x^\varepsilon$ and $y = z + \mathbf{x}_{k+1} - T\mathbf{x}_k$), we deduce that on $f(\mathbf{x}_k) > M + f(x^\varepsilon)$ (which implies $\|\mathbf{x}_k^\varepsilon\| > M$),

$$\text{(23)} \qquad \|\mathbf{x}_{k+1}^\varepsilon\| I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq K) \leq$$

$$\leq (g(\mathbf{x}_{k+1}^\varepsilon) + \varepsilon\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|/(V+1)) I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq K),$$

where $g = f_{T\mathbf{x}_k - x^\varepsilon}$. Note that

$$\|T\mathbf{x}_k - x^\varepsilon\| = g(T\mathbf{x}_k - x^\varepsilon) = g(\mathbf{x}_{k+1} - x^\varepsilon + T\mathbf{x}_k - \mathbf{x}_{k+1}) \leq g(\mathbf{x}^\varepsilon_{k+1}) + \|T\mathbf{x}_k - \mathbf{x}_{k+1}\|$$

and therefore

$$\|\mathbf{x}^\varepsilon_{k+1}\| = \|\mathbf{x}_{k+1} - T\mathbf{x}_k + T\mathbf{x}_k - x^\varepsilon\| \leq g(\mathbf{x}^\varepsilon_{k+1}) + 2\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|. \tag{24}$$

From (23) and (24) we deduce that on $f(\mathbf{x}_k) > M + f(x^\varepsilon)$,

$$\begin{aligned}\|\mathbf{x}^\varepsilon_{k+1}\| \leq & g(\mathbf{x}^\varepsilon_{k+1}) + \varepsilon\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq K)/(V+1) \\ & + 2\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| > K).\end{aligned}$$

As $g = f_{T\mathbf{x}_k - x^\varepsilon}$ and $(\mathbf{x}_k)$ is a T-martingale, $E(g(\mathbf{x}^\varepsilon_{k+1})|\mathcal{F}_k) = g(T\mathbf{x}_k - x^\varepsilon) = \|T\mathbf{x}_k - x^\varepsilon\|$, and, by the nonexpansiveness of T and the triangle inequality, $\|T\mathbf{x}_k - x^\varepsilon\| = \|T\mathbf{x}_k - Tx^\varepsilon + Tx^\varepsilon - x^\varepsilon\| \leq \|\mathbf{x}^\varepsilon_k\| + \alpha + \varepsilon$, and therefore

$$E(g(\mathbf{x}^\varepsilon_{k+1})|\mathcal{F}_k) \leq \|\mathbf{x}^\varepsilon_k\| + \alpha + \varepsilon,$$

and thus, on $f(\mathbf{x}_k) > M + f(x^\varepsilon)$,

$$\begin{aligned} E(\|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k) \leq \|\mathbf{x}^\varepsilon_k\| + \alpha + \varepsilon + \\ + E(\varepsilon\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|/(V+1) + 2\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| I(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| > K)|\mathcal{F}_k),\end{aligned}$$

and therefore using (21) and (22),

$$E(\|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k) - \|\mathbf{x}^\varepsilon_k\|) I(f(\mathbf{x}_k) > M + f(x^\varepsilon)) \leq \alpha + 4\varepsilon.$$

Therefore,

$$\limsup \frac{1}{n}\sum_1^n (E(\|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k) - \|\mathbf{x}^\varepsilon_k\|) I(f(\mathbf{x}_k) > M + f(x^\varepsilon)) \leq \alpha + 4\varepsilon.$$

As $\alpha > 0$, $f(\mathbf{x}_k) \to \infty$ a.e., implying that $\sum_1^\infty I(f(\mathbf{x}_k) \leq M + f(x^\varepsilon))$ is finite a.e. and thus

$$\limsup \frac{1}{n}\sum_1^n E(\|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k) - \|\mathbf{x}^\varepsilon_k\| \leq \alpha + 4\varepsilon. \tag{25}$$

Set $y_{k+1} = \|\mathbf{x}^\varepsilon_{k+1}\| - E(\|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k)$. Note that (y_n) is $\mathcal{F}_n$-adapted and that $E(y_{n+1}|\mathcal{F}_n) = 0$ and thus the sequence $(y_k)_{k=1}^\infty$ is uncorrelated. Also,

$$\begin{aligned}\mathrm{Var}(y_{k+1}) = & E((\|\mathbf{x}^\varepsilon_{k+1}\| - E(\|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k))^2) \\ \leq & 2E((\|\mathbf{x}^\varepsilon_{k+1}\| - \|T\mathbf{x}_k - x^\varepsilon\|)^2) + 2E(E(\|T\mathbf{x}_k - x^\varepsilon\| - \|\mathbf{x}^\varepsilon_{k+1}\||\mathcal{F}_k)^2) \\ \leq & 2E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^2) + 2E(\|T\mathbf{x}_k - \mathbf{x}_{k+1}\|^2) \\ \leq & 4V.\end{aligned}$$

Thus $(\mathrm{Var}(y_k))_{k=1}^{\infty}$ is uniformly bounded and, by the strong law of large numbers for uncorrelated random variables,

$$\lim \frac{1}{n}\sum_{1}^{n}(\|\mathbf{x}_{k+1}^{\varepsilon}\| - E(\|\mathbf{x}_{k+1}^{\varepsilon}\| | \mathcal{F}_n) = 0,$$

which, together with (25), implies that

$$\limsup \frac{1}{n}\sum_{k=1}^{n}(\|\mathbf{x}_{k+1}^{\varepsilon}\| - \|\mathbf{x}_{k}^{\varepsilon}\|) = \limsup \frac{1}{n}(\|\mathbf{x}_{n+1}^{\varepsilon}\| - \|\mathbf{x}_{1}^{\varepsilon}\|) \leq \alpha + 4\varepsilon,$$

and thus $\limsup \|\mathbf{x}_{n+1}^{\varepsilon}\|/n \leq \alpha + 4\varepsilon$. As $|\|\mathbf{x}_n^{\varepsilon}\| - \|\mathbf{x}_n\|| \leq \|x^{\varepsilon}\|$, it follows that $\limsup \|\mathbf{x}_n\|/n \leq \alpha + 4\varepsilon$ and, as this holds for all $\varepsilon > 0, \limsup \|\mathbf{x}_n\|/n \leq \alpha$ a.e., which, together with (19), completes the proof of the theorem for $\alpha > 0$.

If $\alpha = 0$, let $Y = (X \bigoplus R)_2$ (i.e., $\|(x,s)\|^2 = \|x\|^2 + s^2$). We define a nonexpansive map $\hat{T}$: $Y \to Y$ by $\hat{T}(x,t) = (Tx, t+1)$. Let $y_n = (x_n, n)$. Then $\hat{T}$ is nonexpansive, Y is uniformly Fréchet differentiable, and (y_n) is a Y-valued stochastic process with

$$E(y_{n+1}|\mathcal{F}_n) = \hat{T}(y_n),$$

$\|y_{k+1} - Ty_k\| = \|\mathbf{x}_{k+1} - T\mathbf{x}_k\|$, and $\inf\{\|\hat{T}y - y\| : y \in Y\} = 1$. Thus, by the result already proved in the case $\alpha > 0$, $\lim \|y_n\|/n = 1$ a.e., which means that $\lim(\|x_n\|^2 + n^2)^{\frac{1}{2}}/n) = 1$, which implies that $\lim_{n\to\infty} \|x_n\|/n = 0$. This completes the proof of Theorem 2. ∎

Remark: If X is finite dimensional, then X is uniformly smooth iff the norm of X is smooth, i.e., differentiable at each $x \neq 0$ and X is strictly convex iff the norm of the dual X^* is Fréchet differentiable.

In the finite dimensional case we will prove converses to the convergence results. The direct and converse statements are summarized in the two theorems below.

THEOREM 3: *The following conditions on the finite dimensional normed space* $(X, \| \ \|)$ *are equivalent:*

(i) *For every nonexpansive* T: $X \to X$ *and every* T*-martingale* $(\mathbf{x}_n)_{n=0}^{\infty}$ *with* $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|$ *uniformly bounded,* $\|x_n\|/n$ *converges a.e.*

(ii) *For every nonexpansive* T: $X \to X$ *and every* T*-martingale* $(\mathbf{x}_n)_{n=0}^{\infty}$, *with* $E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^2 \mid \mathcal{F}_k)$ *uniformly bounded,* $\|\mathbf{x}_n\|/n$ *coverges a.e.*

(iii) *The norm of* X *is smooth.*

THEOREM 4: *The following conditions on the finite dimensional normed space* $(X, \| \ \|)$ *are equivalent:*

(i) *For every nonexpansive* $T : X \to X$ *and every* T*-martingale* $(\mathbf{x}_n)_{n=0}^{\infty}$, *with* $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|$ *uniformly bounded,* $\lim \mathbf{x}_n/n$ *coverges a.e.*

(ii) *For every nonexpansive* $T : X \to X$ *and every* T*-martingale* $(\mathbf{x}_n)_{n=0}^{\infty}$, *with* $E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p \mid \mathcal{F}_k)$ *uniformly bounded for some* $1 < p \leq 2$, $\lim \mathbf{x}_n/n$ *coverges a.e.*

(iii) *The norm of* X *is strictly convex and smooth.*

A finite-dimensional space is uniformly smooth iff its norm is Fréchet differentiable; thus the implications (iii) $\to$ (ii) of Theorems 3 and 4 follow from Theorem 2. The implication (ii) $\to$ (i) is obvious. By Theorem 1.4 of [5], condition (i) of Theorem 4 implies that the norm of X is strictly convex. It remains to show the implication (i) $\to$ (iii) in Theorem 3. We will show that if X is a Banach space whose norm is not Fréchet differentiable, then there is a nonexpansive map T: $X \to X$ and a T-martingale $(\mathbf{x}_n)_{n=0}^{\infty}$ with $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq 1$ everywhere and for which $\limsup \|\mathbf{x}_n\|/n \neq \liminf \|\mathbf{x}_n\|/n$ a.e.

Assume that the norm is not Fréchet differentiable at a given x in $S(X)$. There is an $\varepsilon > 0$ and a measurable function $t \to y_t$ from $(0, \infty)$ to $S(X)$ such that for all $0 < t < \infty$,

$$\frac{\|tx + y_t\| + \|tx - y_t\|}{2} > t + \varepsilon.$$

Define T: $X \to X$ by $Ty = (\|y\| + 1)x$. Then $\|Ty - Tz\| \leq |\|y\| - \|z\|| \leq \|y - z\|$, i.e., T is nonexpansive. We define a T-martingale $(\mathbf{x}_k)$ taking on values in X as follows: Let n_i be an increasing sequence of positive integers with $n_1 = 1$ and $\lim_{i\to\infty} n_{i+1}/n_i = \infty$. For every $n_{2i-1} \leq k < n_{2i}$,

$$\mathbf{x}_k = T\mathbf{x}_{k-1}$$

where $\mathbf{x}_0 = 0$, and for $n_{2i} \leq k < n_{2i+1}$,

$$E(I(\mathbf{x}_k = T\mathbf{x}_{k-1} + y_k) \mid \mathcal{F}_{n-1}) = 1/2 = E(I(\mathbf{x}_n = T\mathbf{x}_{k-1} - y_k) \mid \mathcal{F}_{n-1}).$$

It is easy to verify that $(\mathbf{x}_k)$ is a T-martingale with $\|\mathbf{x}_{k+1} - T\mathbf{x}_k\| \leq 1$ and that $\|Ty - Tz\| \leq |\|y\| - \|z\|| \leq \|y - z\|$, i.e., T is nonexpansive. Also, $\limsup \|\mathbf{x}_n\|/n \geq 1 + \varepsilon/2$ a.e., while $\liminf \|\mathbf{x}_n\|/n = 1$ a.e.

We do not know whether—under the assumptions of Theorem 1 (nonexpansiveness of T and p-uniform smoothness of X, $1 < p \leq 2$)—the equalities (2) imply condition (1). However, we have a partial result in this direction:

A Banach space X is *p-smooth*, $1 < p \leq 2$, if $\forall x \in S(X)$, $\exists C_x > 0$ s.t. $\forall y \in S(X)$,

$$\|x + ty\| + \|x - ty\| - 2 \leq C_x t^p.$$

If X is not p-smooth — and therefore, obviously, not p-uniformly smooth — there is a nonexpansive $T\colon X \to X$ and a T-martingale $(\mathbf{x}_k)$ with

$$\sum_{k=1}^{\infty} k^{-p} E(\|\mathbf{x}_{k+1} - T\mathbf{x}_k\|^p) < \infty$$

and for which $\liminf \|\mathbf{x}_n\|/n < \limsup \|\mathbf{x}_n\|/n$ a.e.

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